OSCILLATION CRITERIA FOR SECOND-ORDER DELAY DIFFERENTIAL EQUATIONS

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ABSTRACT. The aim of this paper is to establish some new oscillation criteria for the second order retarded differential equation

$$\left(r(t)|u'(t)|^{\alpha-1}u'(t)\right)' + p(t)|u[\tau(t)]|^{\alpha-1}u[\tau(t)] = 0.$$

The results obtained essentially improve known results in the literature.

INTRODUCTION

In this paper we study oscillatory properties of the retarded functional differential equation

$$\left(r(t)|u'(t)|^{\alpha-1}u'(t)\right)' + p(t)|u[\tau(t)]|^{\alpha-1}u[\tau(t)] = 0 \qquad (E_1)$$

under the following hypothesis (H):

(H1) α is a positive number;
(H2) $r(t) \in C^1(t_0, \infty)$, $r(t) > 0$; $R(t) := \int_{t_0}^{t} r^{-\frac{1}{\alpha}}(s)\, ds \to \infty$ as $t \to \infty$;
(H3) $p(t) \in C(t_0, \infty)$, $p(t) > 0$;
(H4) $\tau(t) \in C^1(t_0, \infty)$, $\tau(t) \le t$, $lim_{t\to\infty}\tau(t) = \infty$.

By a solution of (E_1) we mean a function $u \in C^1[T_u, \infty)$, $T_u \ge t_0$, which has the property $r(t)|u'(t)|^{\alpha-1}u'(t) \in C^1[T_u, \infty)$ and satisfies (E_1) on T_u, ∞. We consider only those solutions $u(t)$ of (E_1) which satisfy $\sup\{|u(t) : t \le T|\} > 0$ for all $T \le T_u$. We assume that (E_1) possesses such a solution. A nontrivial solution of (E_1) is said to be oscillatory it if has arbitrarily large zeros; otherwise it is called nonoscillatory. Equation (E_1) is oscillatory if all of its solutions are oscillatory.

Recently, Mirzov[8,9,10], Elbert [3,4], Kusano et al [5,6,7], Chern et al [2], Agarwal et al[1] have observed some similar properties between equation $(E_1)$and the corresponding linear equation

$$\left(r(t)y'\right)' + q(t)y\left(\tau(t)\right) = 0.$$

In this paper we shall continue in this direction the study of oscillatory properties of (E_1). The purpose of this paper is to improve the above-mentioned results. We shall

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establish some new oscillatory criteria for (E_1) and for the following partial case of (E_1)

$$\Big(|u'(t)|^{\alpha-1}u'(t)\Big)' + p(t)|u(\tau(t))|^{\alpha-1}u[\tau(t)] = 0. \tag{E_2}$$

As is customary, all functional inequalities are assumed to hold eventually, that is for all large t.

Main Results

First consider the case where $\alpha \ge 1$.

Theorem 1. *Let $\alpha \ge 1$. Assume that for some $k \in (0,1)$*

$$\int^{\infty} \left(R^{\alpha}[\tau(t)]p(t) - \frac{\alpha\tau'(t)}{4kR[\tau(t)]r^{1/\alpha}[\tau(t)]}\right)dt = \infty \tag{1}$$

Then Eq.(E_1) is oscillatory.

Proof. Assume the converse. Let $u(t)$ be a nonoscillatory solution of (E_1). Without loss of generality we may assume that $u(t) > 0$. This implies

$$\Big(r(t)|u'(t)|^{\alpha-1}u'(t)\Big)' = -p(t)(u[\tau(t)])^{\alpha} < 0$$

Hence, the function $r(t)|u'(t)|^{\alpha-1}u'(t)$ is decreasing and therefore we shall consider the following two cases

(i) $u'(t) > 0$,
(ii) $u'(t) < 0$.

But the condition (ii) implies that for some positive constant M

$$r(t)|u'(t)|^{\alpha-1}u'(t) \le -M < 0.$$

That is

$$-u'(t) \ge \left(\frac{M}{r(t)}\right)^{1/\alpha}.$$

Integrating the above inequality from t_1 to t, we obtain

$$u(t) \le u(t_1) - M^{1/\alpha}(R(t) - R(t_1)).$$

Letting $t \to \infty$, in the above inequality and using (H2), we get $u(t) \to -\infty$. This contradiction proves that (i) holds. On the other hand, using the fact that $[r(t)(u'(t))^{\alpha}]^{1/\alpha}$ is nonincreasing, we see that for any $k_1 \in (0,1)$ and all large t

$$\begin{aligned} u[\tau(t)] &\ge \int_{t_1}^{\tau(t)} u'(s)\,ds = \int_{t_1}^{\tau(t)} \frac{1}{r^{1/\alpha}(s)}\Big(r^{1/\alpha}(s)u'(s)\Big)\,ds \\ &\ge r^{1/\alpha}[\tau(t)]u'[\tau(t)]\Big(R[\tau(t)] - R(t_1)\Big) > k_1R[\tau(t)]r^{1/\alpha}[\tau(t)]u'[\tau(t)]. \end{aligned} \tag{2}$$

Define

$$w(t) = R^{\alpha}[\tau(t)]\frac{r(t)(u'(t))^{\alpha}}{(u[\tau(t)])^{\alpha}}. \tag{3}$$

Then $w(t) > 0$ and

$$w'(t) = \frac{\alpha \cdot \tau'(t) R^{\alpha-1}[\tau(t)]}{r^{1/\alpha}[\tau(t)]} \cdot \frac{r(t)\big(u'(t)\big)^{\alpha}}{\big(u[\tau(t)]\big)^{\alpha}} - R^{\alpha}[\tau(t)]p(t)$$
$$- \alpha R^{\alpha}[\tau(t)] \frac{r(t)\big(u'(t)\big)^{\alpha}}{\big(u[\tau(t)]\big)^{\alpha+1}} u'[\tau(t)]\tau'(t). \tag{4}$$

Taking into account (2) and the monotonicity of $r(t)(u'(t))^{\alpha}$, we conclude that

$$\frac{u'[\tau(t)]}{u[\tau(t)]} = \frac{1}{r[\tau(t)]} \cdot \frac{r[\tau(t)]\big(u'[\tau(t)]\big)^{\alpha}}{\big(u[\tau(t)]\big)^{\alpha}} \cdot \left(\frac{u[\tau(t)]}{u'[\tau(t)]}\right)^{\alpha-1}$$
$$\geq \frac{r(t)\big(u'(t)\big)^{\alpha}}{u[\tau(t)]^{\alpha}} \cdot \frac{k R^{\alpha-1}[\tau(t)]}{r^{1/\alpha}[\tau(t)]},$$

where $k = k_1^{\alpha-1} \in (0,1)$. Combining the previous inequality with (4) we get

$$w'(t) \leq \frac{\alpha\tau'(t)}{R[\tau(t)]r^{1/\alpha}[\tau(t)]} w(t) - \frac{k\alpha\tau'(t)}{R[\tau(t)]r^{1/\alpha}[\tau(t)]} w^2(t) - R^{\alpha}[\tau(t)]p(t).$$

Since the polynomial $P(w) = w - kw^2 \leq 1/(4k)$, the previous inequality implies

$$w'(t) \leq \frac{\alpha\tau'(t)}{4R[\tau(t)]r^{1/\alpha}[\tau(t)]} - R^{\alpha}[\tau(t)]p(t).$$

Integrating this estimate from t_1 to t, we have

$$w(t) \leq w(t_1) - \int_{t_1}^{t} \left[R^{\alpha}[\tau(t)]p(s) - \frac{\alpha\tau'(s)}{4R[\tau(s)]r^{1/\alpha}[\tau(s)]}\right] ds .$$

Letting $t \to \infty$ we get in view of (1) that $w(t) \to -\infty$. This contradiction completes the proof.

Corollary 1. *Let $\alpha \geq 1$. Assume that*

$$\liminf_{t\to\infty} \frac{R^{\alpha+1}[\tau(t)]r^{1/\alpha}[\tau(t)]p(t)}{\tau'(t)} > \frac{\alpha}{4}. \tag{5}$$

Then Eq.(E_1) is oscillatory.

Proof. It is not hard to verify that (5) yields the existence of $k \in (0,1)$ and $\epsilon > 0$ such that for all large t

$$\frac{R^{\alpha+1}[\tau(t)]r^{1/\alpha}[\tau(t)]p(t)}{\tau'(t)} > \frac{\alpha}{4k} + \epsilon .$$

This means that

$$R^{\alpha}[\tau(t)]p(t) - \frac{\alpha\tau'(t)}{4kR[\tau(t)]r^{1/\alpha}[\tau(t)]} > \epsilon \frac{\tau'(t)}{R[\tau(t)]r^{1/\alpha}[\tau(t)]} . \tag{6}$$

Now, it is obvious that (6) implies (1) and the assertion of this corollary follows from Theorem 1.

Imposing stronger condition on the function $r(t)$ we are able to prove another oscillation criterion for Eq.(E_1).

Theorem 2. *Let $r'(t) > 0$ and $\alpha \geq 1$. Assume that for some $k \in (0,1)$*

$$\int^{\infty} \left(R^{\alpha}[\tau(t)]p(t) - \frac{\alpha\tau'(t)R^{\alpha-2}[\tau(t)]r^{1-\frac{2}{\alpha}}[\tau(t)]}{4k\tau^{\alpha-1}(t)} \right) dt = \infty. \tag{8}$$

Then Eq.(E_1) is oscillatory.

Proof. Assume that $u(t)$ is an eventually positive solution of (E_1). From the proof of Theorem 1 we know that $u'(t) > 0$ and that $r(t)\big(u'(t)\big)^{\alpha}$ is decreasing. Moreover, since

$$0 > \left(r(t)\big(u'(t)\big)^{\alpha}\right)' = r'(t)\big(u'(t)\big)^{\alpha} + \alpha r(t)\big(u'(t)\big)^{\alpha-1}u''(t)\ ,$$

we see that $u''(t) < 0$. It is easy to verify that for any $k_1 \in (0,1)$ and all large t

$$u[\tau(t)] \geq \int_{t_1}^{\tau(t)} u'(s)ds \geq u'[\tau(t)]\big(\tau(t) - t_1\big) \geq k_1\tau(t)u'[\tau(t)]\ . \tag{9}$$

Let $w(t)$ be defined as in (3). Then $w(t) > 0$ and (4) holds. Using (9) and the monotonicity of $r(t)(u'(t))^{\alpha}$ we conclude that

$$\begin{aligned} \frac{u'[\tau(t)]}{u[\tau(t)]} &= \frac{1}{r[\tau(t)]} \cdot \frac{r[\tau(t)]\big(u'[\tau(t)]\big)^{\alpha}}{\big(u[\tau(t)]\big)^{\alpha}} \cdot \left(\frac{u[\tau(t)]}{u'[\tau(t)]}\right)^{\alpha-1} \\ &\geq \frac{1}{r[\tau(t)]} \cdot \frac{r(t)\big(u'(t)\big)^{\alpha}}{\big(u[\tau(t)]\big)^{\alpha}} \big(k_1\tau(t)\big)^{\alpha-1}. \end{aligned}$$

Using this estimate in (3), we have

$$w'(t) \leq \frac{\alpha\tau'(t)}{R[\tau(t)]r^{1/\alpha}[\tau(t)]}w(t) - \frac{\alpha k\tau'(t)\tau^{\alpha-1}(t)}{R^{\alpha}[\tau(t)]r[\tau(t)]}\, w^2(t) - R^{\alpha}[\tau(t)]p(t)\ ,$$

where $k = k_1^{\alpha-1} \in (0,1)$. It is easy to check that

$$\begin{aligned} w'(t) \leq\ & \frac{\alpha\tau'(t)R^{\alpha-2}[\tau(t)]r^{1-\frac{2}{\alpha}}[\tau(t)]}{4k\tau^{\alpha-1}(t)} - R^{\alpha}[\tau(t)]p(t) \\ & - \frac{\alpha k\tau'(t)\tau^{\alpha-1}(t)}{R^{\alpha}[\tau(t)]r[\tau(t)]}\left[w(t) - \frac{R^{\alpha-1}[\tau(t)]r^{1-\frac{1}{\alpha}}[\tau(t)]}{2k\tau^{\alpha-1}(t)}\right]^2 \end{aligned}$$

Consequently,

$$w'(t) \leq \frac{\alpha\tau'(t)R^{\alpha-2}[\tau(t)]r^{1-\frac{2}{\alpha}}[\tau(t)]}{4k\tau^{\alpha-1}(t)} - R^{\alpha}[\tau(t)]p(t)\ .$$

Now, we can proceed exactly as in the proof of Theorem 1 to obtain desirable contradiction with the positivity of $w(t)$. So, this part of the proof can be omitted.

Remark 1. Conditions (1) in Theorem 1 and (8) of Theorem 2 are equivalent for $\alpha = 1$. Moreover, it can be easily checked that for $\alpha = 1$ we can let $k = 1$ in (1) and (8), respectively.

Remark 2. Theorems 1 and 2 improve Theorem 1 in [2] and Theorem 1 improves Theorem 2.3 in [1].

The conclusions of Theorems 1 and 2 lead to the following results for Eq.(E_2).

Theorem 3. *Let $\alpha \geq 1$. Assume that for some $k \in (0,1)$*

$$\int^{\infty} \left(\tau^{\alpha}(t)p(t) - \frac{\alpha\tau'(t)}{4k\tau(t)} \right) dt = \infty \,.$$

Then Eq.(E_2) is oscillatory.

Proof. Assuming the converse, we admit that $x(t)$ is an eventually positive solution of (E_2). Following the steps of the proof of Theorem 1 (or Theorem 2) and setting

$$w(t) = \left(\frac{\tau(t)u'(t)}{u[\tau(t)]} \right)^{\alpha}$$

we get a desirable contradiction.

Corrolary 1 leads to the following:

Corollary 2. *Let $\alpha \geq 1$. Assume that*

$$\liminf_{t\to\infty} \frac{\tau^{\alpha+1}(t)p(t)}{\tau'(t)} > \frac{\alpha}{4} \,.$$

Then Eq.(E_2) is oscillatory.

Now consider the case where $0 < \alpha < 1$.

Theorem 4. *Let $0 < \alpha < 1$. Assume that*

$$\int^{\infty} \left(R^{\alpha}[\tau(t)]p(t) - \frac{\alpha\tau'(t)}{4R^{2-\alpha}[\tau(t)]r^{\frac{2}{\alpha}-1}[\tau(t)]\tilde{P}(t)} \right) dt = \infty, \tag{10}$$

where

$$\tilde{P}(t) = \left(\frac{1}{r[\tau(t)]} \int_t^{\infty} p(s)ds \right)^{\frac{1-\alpha}{\alpha}} .$$

Then Eq.(E_1) is oscillatory.

Proof. It can be assumed that Eq.(E_1) has an eventually positive solution $u(t)$. Then, exactly as in the proof of Theorem 1 we conclude that $u'(t) > 0$ and moreover $r(t)(u'(t))^{\alpha}$ is decreasing. Using these facts and integrating (E_1) from t to ∞ we have

$$r(\tau(t))\left(u'(\tau(t))\right)^{\alpha} \geq r(t)(u'(t)) \geq \int_{t_1}^{\infty} p(s)u^{\alpha}[\tau(s)]ds \geq u^{\alpha}[\tau(t)] \int_t^{\infty} p(s)ds.$$

Thus

$$\left(\frac{u'(\tau(t))}{u(\tau(t))} \right)^{1-\alpha} \geq \tilde{P}(t) \,.$$

Defining $w(t)$ as in (3), we see that (4) holds. I can easily be checked that

$$\frac{u'[\tau(t)]}{u[\tau(t)]} = \frac{1}{r[\tau(t)]} \cdot \frac{r[\tau(t)](u'[\tau(t)])^{\alpha}}{(u[\tau(t)])^{\alpha}} \cdot \left(\frac{u'[\tau(t)]}{u[\tau(t)]} \right)^{1-\alpha} \geq \frac{\tilde{P}(t)}{r[\tau(t)]} \cdot \frac{r(t)(u'(t))^{\alpha}}{(u[\tau(t)])^{\alpha}}$$

Combining this with (4) we have

$$w'(t) \leq \frac{\alpha\tau'(t)}{R[\tau(t)]r^{1/\alpha}[\tau(t)]}w(t) - \frac{\alpha\tau'(t)\tilde{P}(t)}{R^\alpha[\tau(t)]r[\tau(t)]}w^2(t) - R^\alpha[\tau(t)]p(t)\ .$$

Direct computation shows that

$$w'(t) \leq \frac{\alpha\tau'(t)R^{\alpha-2}[\tau(t)]r^{1-\frac{2}{\alpha}}[\tau(t)]}{4\tilde{P}(t)} - R^\alpha[\tau(t)]p(t)$$
$$- \frac{\alpha\tau'(t)\tilde{P}(t)}{R^\alpha[\tau(t)]r[\tau(t)]}\left[w - \frac{R^{\alpha-1}[\tau(t)]r^{1-\frac{1}{\alpha}}[\tau(t)]}{2\tilde{P}(t)}\right]^2$$

Therefore,

$$w'(t) \leq \frac{\alpha\tau'(t)}{4R^{2-\alpha}[\tau(t)]r^{\frac{2}{\alpha}-1}[\tau(t)]\tilde{P}(t)} - R^\alpha[\tau(t)]p(t)\ .$$

Integrating this estimate from t to ∞ we obtain in view of (10) that $\lim_{t\to\infty} w(t) = -\infty$. This contradiction proves the theorem.

For the partial case of (E_1) we immediately have the following

Theorem 5. *Let* $0 < \alpha < 1$. *Denote* $\tilde{P}_1(t) = \left(\int_t^\infty p(s)ds\right)^{\frac{1-\alpha}{\alpha}}$. *If*

$$\int^\infty \left(\tau^\alpha(t)p(t) - \frac{\alpha\tau'(t)}{4\tau^{2-\alpha}(t)\tilde{P}_1(t)}\right) dt = \infty, \tag{11}$$

then Eq.(E_2) *is oscillatory.*

The proof of this theorem is similar to the proof of Theorem 2 and therefore is omitted.

Examples

Example 1. Consider

$$\left[|u'(t)|^{\alpha-1}u'(t)\right]' + \frac{a}{t^{\alpha+1}}|u[\lambda t]|^{\alpha-1}u[\lambda t] = 0, \quad 0 < \lambda < 1, \quad \alpha \geq 1, \quad a > 0 \tag{12}$$

If

$$\frac{\lambda^\alpha}{\alpha}a > \frac{1}{4}$$

then, from Corrolary 2, it follows that Eq.(12) is oscillatory. Observe that our condition essentially improves the condition

$$\frac{\lambda^\alpha}{\alpha}a > 1$$

given in Theorem 2.3 in [1].

On the other hand, the corresponding Theorems 1 and 5 in [2] fail for Eq.(12).

Example 2. For the retarded differential equation

$$\left(|u'(t)|u'(t)\right)' + \frac{a}{t^2}u^2[\sqrt{t}]\,\mathrm{sgn}\,u[\sqrt{t}] = 0, \quad a > 0 \tag{13}$$

Corollary 2 implies oscillation of (13) if

$$a > 1/4.$$

Observe however that Theorem 2.3 in [1] requires the stronger condition

$$a > 2.$$

Also Theorem 1 and 5 in [2] cannot be applied to (13).

Example 3. Consider Eq.(12) with $0 < \alpha < 1$. If

$$\frac{\lambda^\alpha}{\alpha}a > \frac{1}{4^\alpha}$$

then, by Theorem 5, all solutions of this equation oscillate. Observe that this condition essentially improves the condition

$$\frac{\lambda^\alpha}{\alpha}a > 1$$

given in Theorem 2.3 in [1]. Moreover Theorems 1 and 5 in [2] fail for this equation.

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